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Shock waves in a gas with several relaxing internal energy modes

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The structure of plane steady shock waves in a gas with several internal energy modes which relax in parallel is investigated. Transport effects are neglected. Conditions for continuity and monotonicity of the velocity profile are discussed; when all modes have constant specific heats and relaxation times it is established that velocity must decrease monotonically. Internal mode energy contents may overshoot their local equilibrium values.

Numerical results for waves in a hypothetical gas with two relaxing modes are presented for purposes of illustration.

1. Introduction

In general any gas has a number of relaxing internal energy modes whose excitation will exert an often dominant influence on either a part or on the whole of the structure of a compression wave. For the case of one relaxing mode, the situation is well understood and is in some cases amenable to analysis (e.g. Lighthill 1956, where the concepts of the fully and partly dispersed waves are introduced and discussed.) Where more than one relaxing mode is concerned, it is often the case that one relaxation time is extremely long compared with the others. It is then usual to assume that all the short-relaxation-time modes are in equilibrium, thereby reducing the problem to one of the single mode type. Such an assumption is sometimes acceptable for the rotational mode of a diatomic molecule, for example, and any subsequent vibrational relaxations are then treated on the basis of the one-mode model just described (e.g. Johannesen 1961). Not all gases are diatomic of course, and not all relaxation times for the various possible modes of internal energy storage differ by many orders of magnitude. It is therefore of some interest to examine the problem of compression-wave structure in a gas with many relaxing modes, all with different relaxation times. The case of a number of relaxing vibrational modes has been studied before by Blythe (1963). However, an assumption concerning the internal energy behaviour was made which limited the applicability of this work to comparatively strong shocks.

In order to examine the many-mode problem, it is necessary to know how each mode is excited. For molecules with many vibrational modes, there is evidence that one mode may feed from another so that excitation is, at least in part, a series type of process, wherein one mode receives energy by collisions and then passes some on by a purely internal process to remaining modes. We do not

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consider this type of excitation in the present paper; in order not to complicate the situation too much with permutations of possible cross-fertilizations amongst the modes, we make the simplifying assumption of excitation in parallel for all modes. That is to say, we assume that all excitations are collisional and that each mode obeys a relaxation equation of familiar form (see equation (16) below). It is possible that 'linear' relaxation equations of this type are valid only for small departures from equilibrium. Since most of the waves considered below are weak, it is probable that these 'linear' equations are adequate.

We shall also assume that the gas is pure, specifically excluding any possibility of chemical reactions which may drain away molecules of one type or another. Thus the number of relaxing modes remains constant throughout.

To analyse the complete structure of a compression wave would demand the retention of viscous and heat conduction terms in the conservation equations, with attendant analytical or numerical difficulties. Such molecular transport processes have effective relaxation times of the order of one molecular-collision-time interval and, since many of the internal-mode phenomena of interest are characterized by much longer times, we feel some justification for excluding transport effects from the present work. The combined influences of transport effects and rotational and vibrational relaxation have recently been considered by Scala & Talbot (1963), who show amongst other things that one must be cautious about employing the equivalent-bulk-viscosity concept for rotational relaxation when large departures from local equilibrium are involved (see also the remarks in § 6.11, Clarke & McChesney 1964). Scala & Talbot include cross-coupling effects between the modes, using a set of relaxation equations proposed by Herzfeld & Litovitz (1959). One of the results of their (numerical) analysis is that the internal-mode and translational temperatures never cross over in the wave, but we shall see later on that this is not a general result for many-mode shocks, and some internal-mode energies can overshoot their local equilibrium values in the right circumstances. Bearing in mind the possible difficulties associated with rapidly relaxing modes in certain cases, we shall assume for present purposes that they can be treated as a simple addition to the translational energy, yielding a constant 'active-mode' specific heat.

Having made quite a number of (reasonably justifiable) assumptions, we then attempt to keep the subsequent analysis as general as possible. Our aim is to try to elicit as much general information about the velocity, pressure, density, translational-temperature and internal-mode-energy profiles as possible over the whole compression-wave speed range, without going to the lengths of obtaining specific, complete, solutions. In the absence of transport effects, it therefore follows that we must be concerned with predicting the continuity or otherwise of the velocity *vs.* distance curve (considered in § 2) and with the possible appearance of stationary points in this curve (considered in § 3). Whilst concentrating primary attention on the velocity profile, it will be found that the temperature and internal-energy-mode behaviour will emerge as the analysis proceeds.

In conclusion, the results of some numerical solutions are presented in order to exemplify the general conclusions for some special cases. Only plane steady waves are treated.

2. Continuity of velocity profiles

With the assumptions of a one-dimensional steady flow the equations of conservation of mass, momentum and energy in the absence of transport phenomena can be written as

$$\rho u = Q, \tag{1}$$

$$p + Qu = fQ, \tag{2}$$

$$e + p/\rho + \tfrac{1}{2}u^2 = H, \tag{3}$$

respectively. Q, f and H are constants whose values depend on the (arbitrary) conditions in the flow ahead of the wave. p, ρ and u are the pressure, density and velocity, respectively. e is the specific internal energy, and in dealing with a pure gas we shall write

$$e = e_1 + \sum_{\nu=2}^{m} e_\nu, \tag{4}$$

where e_1 is the energy contained in the active states of unit mass of gas, and e_ν is the specific energy contributed by the νth relaxing internal mode. We assume that there are $m-1$ such internal energy modes, as indicated in equation (4).

Noting that

$$e_1 + p/\rho = h_1 = C_{p1}T_1, \tag{5}$$

where h_1 is the active-mode specific enthalpy, T_1 the translational temperature, and C_{p1} the (constant) active-mode specific heat at constant pressure, we can rewrite equation (3) in the form

$$C_{p1}T_1 + \tfrac{1}{2}u^2 = H - \Sigma e_\nu. \tag{6}$$

From now on we whall write Σ to mean $\Sigma_{\nu=2}^{m}$; summation over any other values of ν will be indicated specifically as and when it proves necessary.

The thermal equation of state is

$$p = \rho R T_1, \tag{7}$$

(where R is the relevant gas constant), whence equations (1), (2) and (7) show that

$$p/\rho = RT_1 = (f-u)u. \tag{8}$$

Remembering that

$$C_{p1} - C_{v1} = R, \tag{9}$$

and writing

$$\gamma_1 = C_{p1}/C_{v1}, \tag{10}$$

where C_{v1} is the (constant) active-mode specific heat at constant volume, equation (6) can be manipulated to give

$$(u-\mu_1 f)^2 = \mu_1^2 f^2 + 2\{(\gamma_1-1)/(\gamma_1+1)\}(\Sigma e_\nu - H). \tag{11}$$

We have written

$$\mu_1 \equiv \gamma_1/(\gamma_1+1) \tag{12}$$

for brevity.

At this stage we take note of the physical requirement that each and every $e_\nu, \nu \geqslant 2$, shall be a continuous function of the spatial co-ordinate x. In other words, whilst we admit the possibility that p, ρ, u and T_1 *may* change discontinuously with x, we require that e_ν shall never do so.

With this information, it is clear that the right-hand side of equation (11) is a continuous function of x; whence it follows that

$$(u-\mu_1 f)^2$$

must be a continuous function of x.

Clearly the latter condition is satisfied if $u(x)$ is everywhere continuous, but we observe that it is possible for u to jump from a value $u_a > \mu_1 f$ to a value $u_b < \mu_1 f$, provided that

$$u_a - \mu_1 f = \mu_1 f - u_b. \tag{13}$$

Equation (13) can be recognized as a jump solution of equations (1)–(3) with Σe_ν fixed across the jump. Consequently it represents a Rankine–Hugoniot discontinuity in a frozen flow (the internal modes do not change their energy content on crossing the front), and so the transition can only take place from u_a down to u_b as x increases in the flow direction. The expansion discontinuity, across which u jumps from u_b up to u_a is forbidden by the Second Law of Thermodynamics.

Clearly the speed $\mu_1 f$, which is characteristic of the particular upstream ($x = -\infty$) conditions, is an important quantity. Noting from equations (1) and (2) that

$$f = u + p/\rho u, \tag{14}$$

we observe that the conditions $u \gtrless \mu_1 f$ are equivalent to

$$u^2 \gtrless \gamma_1 p/\rho = a_f^2. \tag{15}$$

a_f is the frozen speed of sound, so that when u is equal to $\mu_1 f$ the local frozen Mach number, u/a_f, is equal to unity.

Although we have shown that a frozen shock (as we may call the discontinuity described by equation (13)) is a possibility, we have yet to decide under what circumstances it is likely to appear, if at all. One criterion must be the value of u_∞ in the equilibrium free stream, for, if $u_\infty < \mu_1 f$ (and hence $u_\infty < a_{f\infty}$), it is clear that no initial frozen shock can occur. Such waves will be called fully dispersed.

When $u_\infty > \mu_1 f$ we can show that the frozen shock must occur at the head of all compression waves, for the following reasons. First of all we must write down the relaxation equations which describe how the internal energy modes are excited. We shall assume that

$$\tau_\nu u \frac{de_\nu}{dx} = e_\nu(T_1) - e_\nu \equiv \Delta e_\nu \tag{16}$$

for all ν. τ_ν is the relevant relaxation time, a positive quantity, although not necessarily constant, and $e_\nu(T_1)$ is the value of the internal energy e_ν when the mode is in complete equilibrium with the active states at the local translational temperature T_1. (The difference Δe_ν is defined in equation (16) for later convenience.) We note that, if two relaxation times, τ_ν and $\tau_{\nu+1}$ for example, are equal everywhere, the two relaxation equations can be added and the total energy $e_\nu + e_{\nu+1}$ considered as the energy of a single mode.

Now equation (8) shows that

$$R\frac{dT_1}{du} = f - 2u. \tag{17}$$

When $u_\infty > \mu_1 f$ there will be a region near to u_∞ for which the condition $u > \mu_1 f$ is also satisfied and, since $\gamma_1 > 1$, it follows that $u > \frac{1}{2}f$ will necessarily be true in that region. From equation (17) it is clear that $dT_1/du < 0$ under these circumstances.

Equation (11) shows that

$$\Sigma \frac{de_\nu}{du} = \left(\frac{\gamma_1+1}{\gamma_1-1}\right)(u-\mu_1 f), \tag{18}$$

whence

$$\Sigma \frac{de_\nu}{du} > 0, \quad u > \mu_1 f. \tag{19}$$

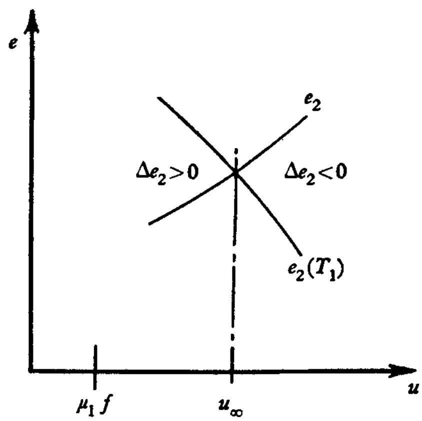

FIGURE 1.

Condition (19) demands that *at least one* of the de_ν/du shall be > 0, and, without loss of generality, we shall assume that

$$de_2/du > 0. \tag{20}$$

The equilibrium value of e_2, namely $e_2(T_1)$, is a monotonically increasing function of T_1, so that if $dT_1/du < 0$, $de_2(T_1)/du < 0$ too. With the proviso that the flow at $x = -\infty$ is in equilibrium, so that

$$e_\nu = e_\nu(T_{1\infty}) \quad \text{or} \quad \Delta e_\nu = 0 \quad \text{at} \quad x = -\infty,$$

we can now sketch the behaviour of $e_2(T_1)$ and e_2 in the vicinity of u_∞ (see figure 1). Let us first suppose that u decreases with increasing x from its upstream value u_∞ (i.e. $du/dx < 0$). Then it is apparent from figure 1 that $\Delta e_2 > 0$, and from equation (16) that $de_2/dx > 0$. Consequently

$$\frac{de_2}{du} = \frac{de_2}{dx}\left(\frac{du}{dx}\right)^{-1} < 0,$$

in violation of the original assumption in equation (20). If $du/dx > 0$, we find $\Delta e_2 < 0$, $de_2/dx < 0$ and $de_2/du < 0$ once again.

Thus a condition like equation (20) cannot be satisfied for any e_ν, and consequently we cannot satisfy condition (19) for any $du/dx \neq 0$. We infer that there can be no continuous variation of $u(x)$ from the value $u_\infty > \mu_1 f$; whence the only accessible route from u_∞ to any other value of u must be through the frozen shock, described (see equation (13)) by the relation

$$u_b(<\mu_1 f) = 2\mu_1 f - u_\infty. \tag{21}$$

Differentiating equation (11) with respect to x gives

$$(u-\mu_1 f)\frac{du}{dx} = \left(\frac{\gamma_1-1}{\gamma_1+1}\right)\Sigma\frac{de_\nu}{dx}. \tag{22}$$

In particular, when $u = u_b$,

$$\left(\frac{du}{dx}\right)_{u=u_b} = -(\mu_1 f - u_b)^{-1}\left(\frac{\gamma_1-1}{\gamma_1+1}\right)\Sigma\left(\frac{de_\nu}{dx}\right)_{u=u_b}, \tag{23}$$

and equation (16) shows that

$$\left(\frac{de_\nu}{dx}\right)_{u=u_b} = \frac{e_\nu(T_{1b})-e_\nu(T_{1\infty})}{u_b\,\tau_{\nu b}}, \tag{24}$$

since e_ν still has its upstream equilibrium value. Equations (8) and (21) show that

$$\frac{1}{2}\frac{R}{f}\left(\frac{\gamma_1+1}{\gamma_1-1}\right)(T_{1b}-T_{1\infty}) = u_\infty - \mu_1 f > 0. \tag{25}$$

Consequently all $de_\nu/dx > 0$ at $u = u_b$ and, from (23), du/dx is always negative at $u = u_b$. We infer that $u(x)$ must subsequently either decrease monotonically to its final equilibrium value or at least pass through a local minimum value if the variations of $u(x)$ are *not* to be monotonic. This latter question will be discussed shortly. Waves for which $u_\infty > \mu_1 f$ will be called partly dispersed.

Having established the initial behaviour of $u(x)$ for $u_\infty > \mu_1 f$, we turn now to a consideration of the initial behaviour for the case $u_\infty < \mu_1 f$. If we rewrite equation (16) in the form

$$\frac{d}{dx}\Delta e_\nu + \frac{1}{u\tau_\nu}\Delta e_\nu = \frac{d}{dx}e_\nu(T_1), \tag{26}$$

a solution for Δe_ν can be written down as follows:

$$\Delta e_\nu = \int_b^x \frac{d}{dy}e_\nu(T_1)\exp\left\{-\int_y^x (dx'/u\tau_\nu)\right\}dy + A_\nu. \tag{27}$$

b is an arbitrarily chosen constant value of x, whilst A_ν is clearly the value of Δe_ν when $x = b$. In the present case ($u_\infty < \mu_1 f$), we choose $b = -\infty$, whence $A_\nu = 0$ since the flow is in equilibrium at $x = -\infty$. Since we can write

$$\frac{de_\nu(T_1)}{dy} = \frac{de_\nu(T_1)}{dT_1}\frac{dT_1}{dy}, \tag{28}$$

it is clear that the behaviour of Δe_ν is intimately associated with that of dT_1/dy ($de_\nu(T_1)/dT_1$ is essentially positive).

We shall suppose that in addition to $u_\infty < \mu_1 f$, u_∞ also satisfies the condition $u_\infty > \frac{1}{2}f$. (A steady compression wave can only exist if u_∞ is greater than the free-stream equilibrium sound speed. The condition $u \gtrless \frac{1}{2}f$ is interpreted in equation (67) below, from which it is clear that $u_\infty > \frac{1}{2}f$ is essential.) From (8) we find that

$$R\frac{dT_1}{dx} = (f-2u)\frac{du}{dx}, \tag{29}$$

so that, if $du/dx \gtrless 0$, $dT_1/dx \lesseqgtr 0$ under these circumstances.

We can now ask whether $u(x)$ can increase monotonically through the value $\mu_1 f$ in a continuous fashion. Any jump increase through this value is forbidden by the Second Law, as we have remarked previously. If it is to do so, equation (22) makes it clear that $\Sigma(de_\nu/dx)$ must be proportional to $u-\mu_1 f$ and consequently

$$\left(\Sigma\frac{de_\nu}{dx}\right)_{u=\mu_1 f} = 0. \tag{30}$$

But, if $u(x)$ increases monotonically, we have just seen that T_1 decreases monotonically. Hence dT_1/dy will be uniformly negative, and equations (28) and (27) show that all $\Delta e_\nu < 0$. Consequently all $de_\nu/dx < 0$, and condition (30) cannot possibly be satisfied. Hence $u(x)$ does not increase monotonically through the value $\mu_1 f$.

It now follows that, if $u(x)$ is to increase at all, it must pass through an initial maximum value ($<\mu_1 f$). This is because in the overall region of compression the final velocity must be less than u_∞. If du/dx is to be locally equal to zero, equation (22) shows that

$$\Sigma\frac{de_\nu}{dx} = 0 \tag{31}$$

must be satisfied at the stationary point. But, by hypothesis, dT_1/dx will be < 0 from $-\infty$ up to this point, so that all $\Delta e_\nu < 0$ (from equation (27) with $b = -\infty$ and $A_\nu = 0$). Thus all $de_\nu/dx < 0$, and condition (31) cannot be satisfied.

The inference is that $u(x)$ can only decrease initially. It will either continue to do so, or pass first through a minimum value as we have previously found to be the case for $u_\infty > \mu_1 f$. Waves for which $u_\infty < \mu_1 f$ will be called fully dispersed.

To summarize the results of this section:

(i) If $u_\infty > \mu_1 f$, u first decreases discontinuously through a frozen shock to a value $u_b < \mu_1 f$; from then on it either decreases monotonically to its final value or passes first through a local minimum.

(ii) If $u_\infty < \mu_1 f$, u first decreases continuously and thence either monotonically to its final equilibrium value, or passes first through a local minimum.

We would point out that these results have been deduced with a minimum of assumptions about the physical character of the relaxing internal energy modes. In particular, both their specific heats and their relaxation times have been required only to be positive quantities and not necessarily constants.

3. The monotonic character of $u(x)$

The question that still remains open after the work of the previous section is whether or not the velocity profile has an initial minimum value. If we can prove that such a minimum is impossible it follows that $u(x)$ is a monotonically decreasing function under all circumstances.

In attempting to settle this issue, some general observations are in order. Bearing in mind the fact that $u < \mu_1 f$ in the regions where $u(x)$ is continuous, we note first from equations (22) that, when du/dx is zero,

$$\Sigma \frac{de_\nu}{dx} = 0. \tag{32}$$

Next we observe from equation (29) that

$$\frac{dT_1}{dx} = 0, \tag{33}$$

when $du/dx = 0$. Differentiating (29) with respect to x shows that

$$R\frac{d^2T_1}{dx^2} = -2\left(\frac{du}{dx}\right)^2 + (f-2u)\frac{d^2u}{dx^2}. \tag{34}$$

Thus

$$R\frac{d^2T_1}{dx^2} = (f-2u)\frac{d^2u}{dx^2}, \tag{35}$$

when $du/dx = 0$.

Once again we shall consider the two cases, $u_\infty \gtrless \mu_1 f$, separately. First let us examine the case $u_\infty < \mu_1 f$. With the assumption, made previously, that $u_\infty > \frac{1}{2}f$ is essential for the appearance of a steady compression wave, let us assume that $du/dx = 0$ occurs *before* u has decreased through the value $\frac{1}{2}f$. Then, since $du/dx < 0$ prior to the position at which the minimum is assumed to have occurred, equation (29) shows that $dT_1/dx > 0$ over the same range of values of x. Since we take (see equation (27) *et seq.*)

$$\Delta e_\nu = \int_{-\infty}^{x} \frac{de_\nu(T_1)}{dT_1}\frac{dT_1}{dy}\exp\left\{-\int_y^x (dx'/u\tau_\nu)\right\}dy \tag{36}$$

for the case $u_\infty < \mu_1 f$, it follows that each $\Delta e_\nu > 0$; thus each $de_\nu/dx > 0$, and condition (32) cannot be satisfied. The inference is that when $u_\infty < \mu_1 f$, $u(x)$ can have no minimum in the range $u_\infty \geqslant u \geqslant \frac{1}{2}f$. What happens if the wave is strong enough to make u fall below the value $\frac{1}{2}f$, we shall discuss shortly.

Turning to the case $u_\infty > \mu_1 f$, let us assume that $u_b > \frac{1}{2}f$. Equation (21) shows that this condition is equivalent to

$$u_\infty < (2\mu_1 - \tfrac{1}{2})f = \frac{1}{2}\frac{3\gamma_1 - 1}{\gamma_1 + 1}f, \tag{37}$$

so that the wave is of restricted strength. If the assumed minimum of $u(x)$ is to occur before u reaches $\frac{1}{2}f$, we know that $du/dx < 0$ and $dT_1/dx > 0$ from u_b down to the assumed stationary value. It is convenient to locate $u = u_b$ where $x = 0$, and the appropriate version of equation (27) is

$$\Delta e_\nu = \int_0^x \frac{de_\nu(T_1)}{dT_1}\frac{dT_1}{dy}\exp\left\{-\int_y^x (dx'/u\tau_\nu)\right\}dy + \Delta e_{\nu b}, \tag{38}$$

where

$$\Delta e_{\nu b} = e_\nu(T_{1b}) - e_\nu(T_{1\infty}) > 0. \tag{39}$$

It follows from the assumptions made above that again $\Delta e_\nu > 0$ for all ν, and that in consequence condition (32) cannot be satisfied. Once more the inference is that $du/dx = 0$ cannot occur before $u = \frac{1}{2}f$.

Indeed equations (36) and (38) make it clear that any particular Δe_ν cannot be zero or negative until dT_1/dx changes sign and becomes negative. In the case of a wave which is sufficiently strong to make u decrease through $\frac{1}{2}f$, a stationary value for $u(x)$ is still possible.

When $u = \frac{1}{2}f$, (34) shows that

$$R\frac{d^2T_1}{dx^2} = -2\left(\frac{du}{dx}\right)^2, \tag{40}$$

so that T_1 is necessarily a maximum at this location. Since dT_1/dx and du/dx must otherwise be zero together, we remark that the first stationary value of T_1 after the maximum at $\frac{1}{2}f$ must be a minimum, and equation (35) confirms that this coincides with the already established necessary minimum in u.

Without having yet answered the question as to whether a minimum in u does or does not occur, we have now at least narrowed the field within which such a possibility remains.

At this stage we sacrifice some of the generality of the foregoing analysis and make the following assumptions about the behaviour of the internal energy modes. First, we assume that the energy content of each mode can be specified by defining an internal-mode temperature T_ν, which differs in general from T_1, and is only equal to it when equilibrium prevails. Then we assume that

$$\frac{de_\nu}{dT_\nu} = C_\nu = \text{const.} \tag{41}$$

for all ν, although each C_ν may be different. Since we deal only with communicable energies it follows that

$$e_\nu = C_\nu T_\nu, \quad e_\nu(T_1) = C_\nu T_1. \tag{42}$$

Secondly, we assume that all the relaxation times τ_ν are constants. The energies of relaxing internal modes are most often non-linear functions of their temperatures T_ν. The linear relationships in equations (42) can be considered as reasonable approximations, applicable within the relatively small temperature ranges appropriate to comparatively weak shock waves. Similarly, one may take the relaxation times to be some suitable constant mean values throughout such waves.

With these assumptions, we can now write equation (36) as

$$T_1 - T_\nu = \int_{-\infty}^{x} \frac{dT_1}{dy}\exp\left(-Y/\tau_\nu\right)dy, \tag{43}$$

where

$$Y \equiv \int_y^x (dx'/u), \tag{44}$$

and (38) as

$$T_1 - T_\nu = \int_0^x \frac{dT_1}{dy}\exp\left(-Y/\tau_\nu\right)dy + T_{1b} - T_{1\infty}. \tag{45}$$

We note that Y in equation (44) is always positive or zero, the latter value occurring only when $y = x$. Thus the exponential in the integrands of (43) and (45) is always $\leqslant 1$. These two equations show that

$$T_\nu - T_{\nu+1} = \int_b^x \frac{dT_1}{dy}\{\exp\left(-Y/\tau_{\nu+1}\right) - \exp\left(-Y/\tau_\nu\right)\}dy, \tag{46}$$

where $b = -\infty$ or 0 appropriately. Without loss of generality, let us assume that

$$\tau_\nu < \tau_{\nu+1} \quad (\nu \geqslant 2). \tag{47}$$

For $x_m > x > b$, where x_m is the position at which the maximum in T_1 occurs, equation (46) shows that $T_\nu > T_{\nu+1}$, because $dT_1/dx > 0$ in this interval. In addition, we observe that T_ν and $T_{\nu+1}$ can only become equal once more (they are equal at $x = b$) after dT_1/dx has become negative. The inference is that, if the T_ν are to exceed T_1 at all, then they will cross over the $T_1(x)$ curve in the order T_ν before $T_{\nu+1}$, $T_{\nu+1}$ before $T_{\nu+2}$, etc., where the word 'before' implies 'at a smaller value of x than'. When T_ν has crossed over T_1, the value of dT_ν/dx will be negative, and these negative values of the internal-mode-temperature gradients will arise in strict order of increasing values of τ_ν.

To return to the general case for a moment, we differentiate (22) with respect to x, giving

$$\left(\frac{du}{dx}\right)^2 + (u-\mu_1 f)\frac{d^2u}{dx^2} = \left(\frac{\gamma_1-1}{\gamma_1+1}\right)\left\{\Sigma\frac{1}{u\tau_\nu}\left[\frac{de_\nu(T_1)}{dT_1}\frac{dT_1}{dx}-\frac{de_\nu}{dx}\right]+\Sigma\Delta e_\nu\frac{d}{dx}\left(\frac{1}{u\tau_\nu}\right)\right\}, \tag{48}$$

where we have used equation (16) to eliminate de_ν/dx from (22). Now τ_ν is, in general, a function of p and T_1 and hence is a function of u. It follows that $d(1/u\tau_\nu)/dx$ is proportional to du/dx. When $du/dx = 0$, equation (48) therefore gives

$$\left(\frac{\gamma_1-1}{\gamma_1+1}\right)\Sigma\frac{1}{\tau_\nu}\frac{de_\nu}{dx} = -u(u-\mu_1 f)\frac{d^2u}{dx^2}. \tag{49}$$

We readily infer that, if the local minimum value of u is to occur, we should expect to find

$$\Sigma\frac{1}{\tau_\nu}\frac{de_\nu}{dx} > 0 \tag{50}$$

at the point in question. It should be emphasized that condition (50) is completely general and in no way depends on the constancy or otherwise of C_ν and τ_ν.

Returning to the specific case for which C_ν and τ_ν *are constant*, we can rewrite the necessary conditions (32) and (50) for the appearance of a minimum in u as follows:

$$\Sigma C_\nu\frac{dT_\nu}{dx} = 0; \quad \Sigma\frac{C_\nu}{\tau_\nu}\frac{dT_\nu}{dx} > 0. \tag{51}$$

We take a simple case first; assume that only T_2 has crossed T_1, so that we have

$$\frac{dT_2}{dx} < 0; \qquad \frac{dT_\nu}{dx} > 0, \quad (\nu \geqslant 3); \quad \tau_2 < \tau_\nu \qquad (\nu \geqslant 3). \tag{52}$$

Eliminating dT_2/dx from the second of equations (51) with the aid of the first we find that we should have

$$\sum_{\nu=3}^{m} C_\nu\left(\frac{1}{\tau_\nu}-\frac{1}{\tau_2}\right)\frac{dT_\nu}{dx} > 0. \tag{53}$$

Clearly conditions (52) and (53) are incompatible, and no minimum in $u(x)$ is possible in these circumstances. To prove that a minimum is impossible in all circumstances when C_ν and τ_ν are constant, we can proceed as follows. From the inequalities (47) we can write

$$\tau_2^{-1} = \tau_3^{-1} + \epsilon_3 = \tau_4^{-1} + \epsilon_3 + \epsilon_4 = \dots = \tau_m^{-1} + \epsilon_3 + \dots + \epsilon_m, \tag{54}$$

where $\epsilon_n > 0$, $m \geqslant n \geqslant 3$. Then equation (53) which is generally valid, can be written in the form

$$\sum_{\nu=3}^{m} (\epsilon_3 + \epsilon_4 + \dots + \epsilon_m) C_\nu \frac{dT_\nu}{dx} = \sum_{q=3}^{m} \epsilon_q \sum_{\nu=q}^{m} C_\nu \frac{dT_\nu}{dx} < 0. \tag{55}$$

We now assume that

$$\frac{dT_\nu}{dx} < 0, \quad 2 \leqslant \nu \leqslant \tilde{p}; \qquad \frac{dT_\nu}{dx} > 0, \quad \tilde{p}+1 \leqslant \nu \leqslant m; \tag{56}$$

and rewrite (55) as

$$\sum_{q=3}^{\tilde{p}} \epsilon_q \sum_{\nu=q}^{m} C_\nu \frac{dT_\nu}{dx} + \sum_{q=\tilde{p}+1}^{m} \epsilon_q \sum_{\nu=q}^{m} C_\nu \frac{dT_\nu}{dx} < 0. \tag{57}$$

But the first of equations (51) shows that

$$\epsilon_q \sum_{\nu=q}^{m} C_\nu \frac{dT_\nu}{dx} = -\epsilon_q \sum_{\nu=2}^{q-1} C_\nu \frac{dT_\nu}{dx},$$

and putting this result into the first term of (57), we require to find

$$-\sum_{q=3}^{\tilde{p}} \epsilon_q \sum_{\nu=2}^{q-1} C_\nu \frac{dT_\nu}{dx} + \sum_{q=\tilde{p}+1}^{m} \epsilon_q \sum_{\nu=q}^{m} C_\nu \frac{dT_\nu}{dx} < 0. \tag{58}$$

In the first term of equation (58) we have $2 \leqslant \nu \leqslant \tilde{p}-1$, so that from (56) all the relevant dT_ν/dx are < 0, and in the second term of (58) $\tilde{p}+1 \leqslant \nu \leqslant m$, so that all the relevant dT_ν/dx are > 0. Since ϵ_q and C_ν are essentially positive, condition (58) cannot ever be satisfied for any value of $\tilde{p}$.

We deduce that when C_ν and τ_ν are constants $u(x)$ must be a monotonically decreasing function from its initial to its final value, whatever the value of u_∞.

It is worth noting that even though $u(x)$ is a monotonically decreasing function in this case (so that ρ and p will increase monotonically) the translational and internal mode temperatures may have local maxima. The maximum in T_1 *must* occur where $u = \frac{1}{2}f$, whilst maxima in T_ν will arise only when they cross the $T_1(x)$ curve and hence occur *after* u has decreased through $\frac{1}{2}f$. When C_ν and τ_ν are constants, we could now make qualitative sketches of the behaviour of all the relevant variables in the shock region. Rather than do this we shall in § 4 give the results of some numerical calculations for the special case of two relaxing modes.

Whilst the case of constant C_ν and τ_ν has now been satisfactorily resolved, we have not yet cleared up the more general (and indeed strictly more practical) case for which these quantities vary with p and T_1. In the simple case of one relaxing mode, it is clear that conditions (32) and (50) are generally incompatible, and the velocity profile can only be of monotonic-decreasing form. With more than one mode, difficulty arises because of the necessity to discuss a strictly local set of conditions (namely conditions (32) and (50)) which nevertheless depend upon the whole previous history of the flow, and in particular on the complete behaviour of C_ν and τ_ν up to the point in question. Taking as an example the case of two modes, conditions (32) and (50) show that we must find

$$\frac{de_2}{dx}+\frac{de_3}{dx}=0;\quad \left(\frac{1}{\tau_3}-\frac{1}{\tau_2}\right)\frac{de_3}{dx}>0,$$

at the minimum in $u(x)$. Taking $de_3/dx>0$, for example, this means that we must find $\tau_3<\tau_2$ locally. Since $de_2/dx<0$, this implies that the mode with the *longest* relaxation time *at the point in question* must have previously overshot its equilibrium state whilst the other mode (with energy e_3) remains below its local equilibrium state despite having, locally, the shortest relaxation time. With constant relaxation times we have just proved that these conditions cannot hold, but, if the τ_ν and C_ν vary, there does not seem to be any reason why they should not arise. We can find no specific proof that they do not, and although we may venture to suggest that the situation is unlikely, it is quite conceivable that it may arise for certain variations of C_ν and τ_ν. Whether the necessary variations of these quantities can arise in any real gas is another question which we must, for the present, leave open. It would appear that each case must be dealt with on its own individual merits.

4. Numerical solutions for two relaxing modes

With the assumptions of constant C_ν and τ_ν, the relaxation equations (16) become

$$\tau_\nu u\frac{dT_\nu}{dx}=T_1-T_\nu\quad(\nu=2,3),\tag{59}$$

and equation (22) can be rewritten in the form

$$\left(\frac{\gamma_1+1}{\gamma_1-1}\right)(u-\mu_1 f)\frac{du}{dx}=C_2\frac{dT_2}{dx}+C_3\frac{dT_3}{dx}.\tag{60}$$

Eliminating T_2, T_3 and T_1 between these equations and (8) we find the following equation satisfied by u alone:

$$A\frac{d^2u}{dx^2}+B\left(\frac{du}{dx}\right)^2+C\frac{du}{dx}+D=0,\tag{61}$$

where

$$\left.\begin{aligned}A&\equiv\tau_2\tau_3uC_{p1}(\mu_1f-u)/\mu_1,\\ B&\equiv\tau_2\tau_3C_{p1}2(\tfrac{1}{2}\mu_1f-u)/\mu_1,\\ C&\equiv f\{\tau_3(C_{p1}+C_2)+\tau_2(C_{p1}+C_3)\}\\ &\quad-2u\{\tau_3(C_{p1}+C_2-\tfrac{1}{2}R)+\tau_2(C_{p1}+C_3-\tfrac{1}{2}R)\},\\ D&\equiv C_{pe}f-u(C_{pe}-\tfrac{1}{2}R)-RH/u,\\ C_{pe}&=C_{p1}+C_2+C_3.\end{aligned}\right\}\tag{62}$$

C_{pe} is the equilibrium specific heat at constant pressure, and H is the 'energy' constant defined in equation (3). Noting that

$$C_{pe}-\tfrac{1}{2}R=\tfrac{1}{2}C_{pe}\frac{\gamma_e+1}{\gamma_e},\quad R=C_{pe}\frac{\gamma_e-1}{\gamma_e},$$

where γ_e is the (constant) ratio of the equilibrium specific heats, we can rewrite the function D in the form

$$D = -\tfrac{1}{2}C_{pe}\frac{\gamma_e+1}{\gamma_e}\frac{1}{u}\left(u^2-\frac{2\gamma_e}{\gamma_e+1}fu+2\frac{\gamma_e-1}{\gamma_e+1}H\right). \tag{63}$$

The gas flow at $x = -\infty$ (where $u = u_\infty$) and at $x = +\infty$ (where $u = u_s$) is in equilibrium, and all gradients of u are zero at these points. Thus D is zero when $x = \pm\infty$, and it follows that the two roots, u_∞ and u_s, of the quadratic expression in brackets in equation (63) are related as follows:

$$u_\infty + u_s = \frac{2\gamma_e}{\gamma_e+1}f, \tag{64}$$

$$u_\infty u_s = 2\frac{\gamma_e-1}{\gamma_e+1}H. \tag{65}$$

Since we can write, from (3), etc.,

$$H = C_{pe}T_{1m}+\tfrac{1}{2}u_m^2 = \frac{\gamma_e}{\gamma_e-1}RT_{1m}+\tfrac{1}{2}u_m^2 = \frac{a_{em}^2}{\gamma_e-1}+\tfrac{1}{2}u_m^2,$$

where m is either ∞ or s and a_e is the equilibrium speed of sound, equation (65) shows that

$$u_\infty u_s = \frac{2}{\gamma_e+1}(a_{em}^2-u_m^2)+u_m^2,$$

or

$$u_\infty - u_s = \frac{2}{\gamma_e+1}\left(\frac{u_\infty^2-a_{e\infty}^2}{u_\infty}\right) = \frac{2}{\gamma_e+1}\left(\frac{a_{es}^2-u_s^2}{u_s}\right), \tag{66}$$

With $u_\infty \geqslant u_s$ it follows that $u_\infty \geqslant a_{e\infty}$ and $u_s \leqslant a_{es}$. The conditions $u \gtrless \tfrac{1}{2}f$, which have occurred quite frequently above, can be re-interpreted, using equation (14) as

$$u^2 \gtrless p/\rho = a_i^2. \tag{67}$$

a_i is the isothermal (or Newtonian) sound speed, and clearly $a_i < a_e < a_f$ since $1 < \gamma_e < \gamma_1$. Because u must decrease monotonically between u_∞ and u_s, it follows that u will pass through $\tfrac{1}{2}f$ whenever $u_s \leqslant \tfrac{1}{2}f$. From equation (64) this will occur whenever

$$u_\infty \geqslant \frac{3\gamma_e-1}{2(\gamma_e+1)}f. \tag{68}$$

The condition $u_\infty \geqslant a_{e\infty}$ can be re-interpreted as

$$u_\infty \geqslant \frac{\gamma_e}{\gamma_e+1}f, \tag{69}$$

and, since $\gamma_e > 1$, we have a range of u_∞ given by

$$\frac{\gamma_e}{\gamma_e+1}f \leqslant u_\infty \leqslant \frac{3\gamma_e-1}{2(\gamma_e+1)}f \tag{70}$$

within which the wave is not strong enough to cause u to 'go isothermally subsonic' (i.e. pass through $\tfrac{1}{2}f$). When $u_\infty < \mu_1 f$, the wave has a continuous variation of u (in other words it is fully dispersed), and we may inquire whether $u_\infty = \mu_1 f$ lies within the range (70). Clearly $\mu_1 f$ is greater than $\gamma_e f/(\gamma_e+1)$ because $\gamma_1 > \gamma_e$, so we need to know whether $\mu_1 f$ is less than $(3\gamma_e - 1)f/2(\gamma_e + 1)$. This condition only occurs if

$$\gamma_e > (3\gamma_1 + 1)/(\gamma_1 + 3). \tag{71}$$

If we write

$$C_{pe} = C_{p1} + \epsilon; \quad C_{ve} = C_{v1} + \epsilon, \tag{72}$$

so that ϵ is the additional specific heat contributed by the relaxing internal modes, we can re-interpret condition (71) as

$$\epsilon < \tfrac{1}{2}(C_{p1} + C_{v1}), \tag{73}$$

and there is a maximum energy content for the internal modes below which u does not become isothermally subsonic in a fully dispersed wave, and therefore no maximum in T_1 can occur in such a wave.

When $u_\infty > \mu_1 f$, the wave starts with a frozen shock, reducing u to the value u_b (given in (21)) discontinuously. In the present case u then decreases monotonically to u_s. The previous paragraph has shown that u_s *may* be greater than $\frac{1}{2}f$, even when $u_\infty > \mu_1 f$, provided that γ_e has the appropriate value, but we should also inquire whether u_b exceeds $\frac{1}{2}f$. From equation (21) this will occur provided that

$$u_\infty < \frac{1}{2}\frac{3\gamma_1 - 1}{\gamma_1 + 1} f. \tag{74}$$

If u_∞ should exceed this value u_b, and hence necessarily u_s, will be below $\frac{1}{2}f$, and it follows that both du/dx and dT_1/dx will be everywhere negative. Such shocks are comparatively strong. For weaker shocks, with u_∞ in the range given by

$$\mu_1 f \leqslant u_\infty \leqslant \frac{1}{2}\frac{3\gamma_1 - 1}{\gamma_1 + 1} f, \tag{75}$$

$T_1(x)$ will have a local maximum and although (dT_1/dx) at $u = u_b$ is necessarily positive in this upstream velocity range, it is possible that T_{1b} may be either greater or less than T_{1s}. These possibilities occur according as to whether

$$u_\infty \gtrless \tfrac{1}{2}f\left\{1 + \frac{2(\gamma_1\gamma_e - 1)}{(\gamma_1 + 1)(\gamma_e + 1)}\right\} \tag{76}$$

respectively. We note that

$$\mu_1 f < \tfrac{1}{2}f\left\{1 + \frac{2(\gamma_1\gamma_e - 1)}{(\gamma_1 + 1)(\gamma_e + 1)}\right\} < \frac{1}{2}\frac{3\gamma_1 - 1}{\gamma_1 + 1} f, \tag{77}$$

because $1 < \gamma_e < \gamma_1$, so that both cases $T_{1b} \lesseqgtr T_{1s}$ will always occur in the velocity range (75).

We must now set out to find numerical solutions for u from equation (61). Once $u(x)$ is known $T_1(x)$ can be found from (8), and T_2 and T_3 from (59); p and ρ will follow from (1) and (2). The constant f provides a natural and suitable velocity characterizing any particular situation, and accordingly we define the following dimensionless variables:

$$v = u/f; \quad y = x/\tau_3 f; \quad \lambda = \tau_2/\tau_3. \tag{78}$$

From equation (61) we find that

$$\bar{A}\frac{d^2v}{dy^2} + \bar{B}\left(\frac{dv}{dy}\right)^2 + \bar{C}\frac{dv}{dy} + \bar{D} = 0, \tag{79}$$

where

$$\left.\begin{aligned}\bar{A} &= \lambda v(1-v/\mu_1),\\ \bar{B} &= 2\lambda(\tfrac{1}{2}-v/\mu_1),\\ \bar{C} &= 1+(C_2/C_{p1})+\lambda(1+C_3/C_{p1})\\ &\quad -2v\{1+(C_2-\tfrac{1}{2}R)/C_{p1}+\lambda[1+(C_3-\tfrac{1}{2}R)/C_{p1}]\},\\ \bar{D} &= \frac{C_{pe}}{C_{p1}}\frac{\gamma_e+1}{2\gamma_e}\frac{(v_\infty-v)(v-v_s)}{v}.\end{aligned}\right\} \tag{80}$$

The result for $\bar{D}$ follows from the fact that we can write D (see (63)) in the form

$$D = -C_{pe}\frac{\gamma_e+1}{2\gamma_e}\frac{(u-u_\infty)(u-u_s)}{u},$$

since u_∞ and u_s are the roots of the quadratic term in brackets in (63).

Defining

$$t = dv/dy, \tag{81}$$

so that

$$\frac{d^2v}{dy^2} = \frac{dt}{dy} = \frac{dt}{dv}t, \tag{82}$$

equation (79) shows that

$$\frac{dt}{dv} = -\frac{\bar{B}t^2+\bar{C}t+\bar{D}}{\bar{A}t}. \tag{83}$$

This first-order equation has singular points at

$$t = 0, \quad v = v_\infty; \qquad t = 0, \quad v = v_s; \tag{84}$$

which clearly represent the respective upstream and downstream equilibrium states. Writing $t = 1/z$ shows that the equation has additional singular points at $v = \mu_1$ (where $\bar{A} = 0$) and $t = \pm\infty$. However, we know that the only continuous solutions for u, and hence for v, arise when $v < \mu_1$, so that we need not concern ourselves with these two points. When $v_\infty < \mu_1$, we need to compute the integral curve which connects the two points (84), whilst when $v_\infty > \mu_1$ we only require to compute that portion of the integral curve which connects the point v_b, t_b with the point v_s, 0. Equation (21) shows that

$$v_b = 2\mu_1 - v_\infty, \tag{85}$$

whilst (23) and (24), etc., enable us to evaluate t_b.

The point v_∞, 0 is a saddle point and v_s, 0 is a degenerate node through which the integral curves can pass along either one of two lines whose slopes are given by

$$\left(\frac{dt}{dv}\right)_{v_s,0} = -\frac{1}{2}\frac{\bar{C}_s}{\bar{A}_s} \pm \left\{\left(\frac{1}{2}\frac{\bar{C}_s}{\bar{A}_s}\right)^2 - \frac{1}{2}\frac{C_{pe}}{C_{p1}}\frac{(\gamma_e+1)(v_\infty-v_s)}{\gamma_e\bar{A}_s v_s}\right\}^{\frac{1}{2}}. \tag{86}$$

$\bar{A}_s$ and $\bar{C}_s$ are $\bar{A}$ and $\bar{C}$ evaluated at $v = v_s$. It is possible, after some tedious algebra, to show that $\bar{C}_s > 0$ for $v_s < \mu_1$ (which is always true), and that the square-root quantity is always positive. Accordingly, the slopes dt/dv at v_s, 0 are both always real and negative. Which of these two integral curve slopes is the slope of the required solution curve is not of direct importance, since the numerical solution is always started from the saddle point v_∞, 0 when $v_\infty < \mu_1$ and from v_b, t_b when $v_\infty > \mu_1$. The numerical procedures then always select the correct curve. However, equation (86) provides a useful check on the numerical accuracy achieved, since it allows direct evaluation of the final slope for purposes of comparison. We remark too that it is possible to evaluate dt/dv at $v_\infty, 0$ and at v_b, t_b for the solution curves, so enabling the numerical work to proceed directly from initial to final phase-plane (or (t, v)-plane) points, without the need for iterative procedures. The advantages of computing over a finite range $v_s \leqslant v \leqslant v_\infty$ (or v_b) in the independent variable are obvious when compared with the infinite range of values of y necessary for direct (v, y)-plane computation. Once the solution curve in the phase plane is known $v(y)$ is calculated at once by simple quadratures.

The numerical integrations were performed on the Ferranti Pegasus Mk II digital machine at the College of Aeronautics. Some examples of the results obtained are shown graphically in figures 2–6. The gas is a hypothetical one for which

$$C_{p1} = \tfrac{7}{2}R, \quad C_{v1} = \tfrac{5}{2}R, \quad C_2 = R = C_3. \tag{87}$$

Each internal mode is therefore treated as if it is classically excited with two degrees of freedom. It follows that

$$\gamma_1 = 7/5, \quad \gamma_e = 11/9. \tag{88}$$

We note that ϵ, defined in (72), is equal to $2R$ in this case. Since $\frac{1}{2}(C_{p1}+C_{v1})$ is equal to $3R$ condition (73), and hence condition (71), is satisfied and no fully dispersed wave in our model gas can be strong enough to make u pass through $\frac{1}{2}f$. Consequently the translational temperature must increase monotonically in our fully dispersed waves.

It is convenient to deal in terms of non-dimensional temperatures, defined as follows:

$$T'_n = RT_n/f^2 \quad (n = 1, 2, 3). \tag{89}$$

Figures 2–6 are graphs of v *vs.* y and T'_n *vs.* y for various values of λ, each figure being for a different shock strength. Shock strength is specified once v_∞ is known, but it is more physically meaningful if quoted in terms of the upstream equilibrium Mach number $M_{e\infty}$, where

$$M_{e\infty} = u_\infty/a_{e\infty}. \tag{90}$$

$M_{e\infty}$ and v_∞ are related as follows:

$$M^2_{e\infty} = \frac{1}{\gamma_e}\left(\frac{v_\infty}{1-v_\infty}\right). \tag{91}$$

Figure 2 shows profiles for a typical fully dispersed wave. The effect on the velocity profiles of increasing τ_2 for a fixed τ_3 (i.e. of increasing λ) is illustrated in figures 2(*a*) and 2(*b*): it is interesting to note that very large differences in these two quantities are necessary before the velocity profile begins to change from a smooth monotonic shape. This can be seen in figure 2(*b*), where the effect of differing relaxation times shows up as a sharp change of slope around $v = v'_s$ only when λ is as large as 1000. v'_s is the downstream (dimensionless) value of the velocity for the given v_∞, calculated on the assumption that one mode is completely frozen (in this case, mode 2). The effective 'equilibrium γ' is γ'_e, equal to 9/7 in this case, and

$$v'_s = \frac{2\gamma'_e}{\gamma'_e+1} - v_\infty. \tag{92}$$

When λ equals 1000, it is perhaps a reasonable approximation to consider the fully dispersed wave as consisting of two consecutive regions, the behaviour in each being dominated by the relaxation of a particular energy mode. This behaviour is not so apparent from the $v(y)$ curves when λ equals 10 and 100 in figure 2 (*b*). Some extra light is shed on this question by the temperature profiles drawn for λ equal to 100 in figure 2 (*d*). From this it is apparent that quite significant changes have occurred in the 'slow' mode 2 before mode 3 can fairly be said to have come into equilibrium with the active states, although to be sure T_1' and T_3' never differ very greatly.

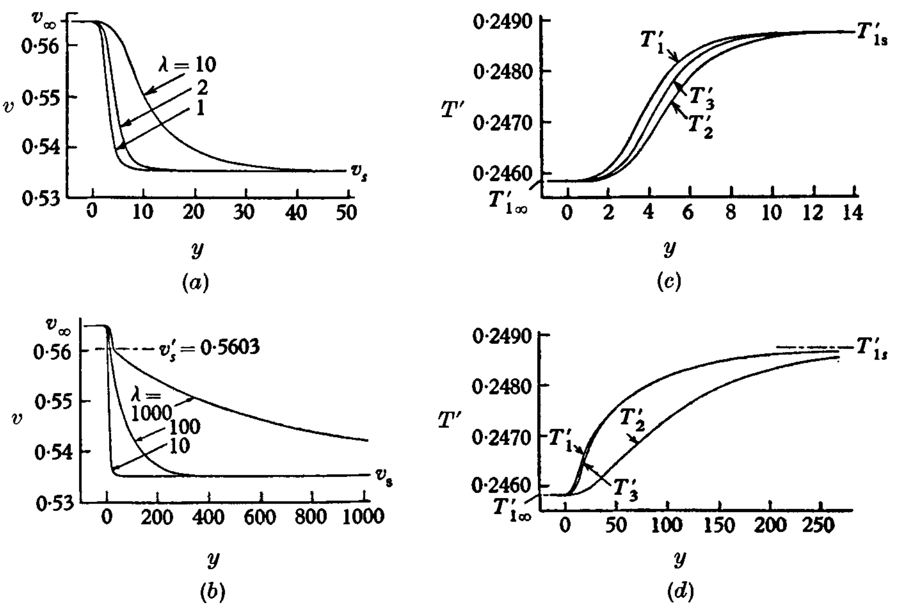

FIGURE 2. Fully dispersed wave. $M_{e\infty} = 1{\cdot}03$; $v_\infty = 0{\cdot}5647$, $v_s = 0{\cdot}5353$; $\gamma_1 = 7/5$, $\gamma_e = 11/9$. $\lambda = \tau_2/\tau_3$. (*a*) Velocity profiles for $\lambda = 1, 2, 10$. (*b*) Velocity profiles for $\lambda = 10, 100, 1000$. (*c*) Temperature profiles for $\lambda = 2$. (*d*) Temperature profiles for $\lambda = 100$.

Figure 3 illustrates the behaviour in a very weak partly dispersed wave, so weak in fact that u_s is still 'isothermally supersonic'. T_1' continues to increase monotonically after its sudden jump at $y = 0$, and in conformity with the deductions made in § 3, there are no overshoots of T_2' or T_3' (see figures 3 (*c*), 3 (*d*)). v_s' in figure 3 (*b*) is calculated from equation (92).

Increasing the strength of the wave leads first to the condition for which a maximum in T_1' occurs in the region of continuous variations, but for which $T_{1b}' < T_{1s}'$. The velocity profiles are similar to those in figures 3 (*a*) and 3 (*b*), with the discontinuity occupying rather more of the full wave amplitude. The temperature profiles, for $\lambda = 2$ only, are exhibited in figure 4, which is drawn to a magnified vertical scale in order to dramatize the significant effects. The overshoot of T_3', the 'fast' mode in this case, is clearly seen, but it should be noted that the amount by which T_3' actually exceeds T_1' is a small fraction of the total variations of temperature which are involved.

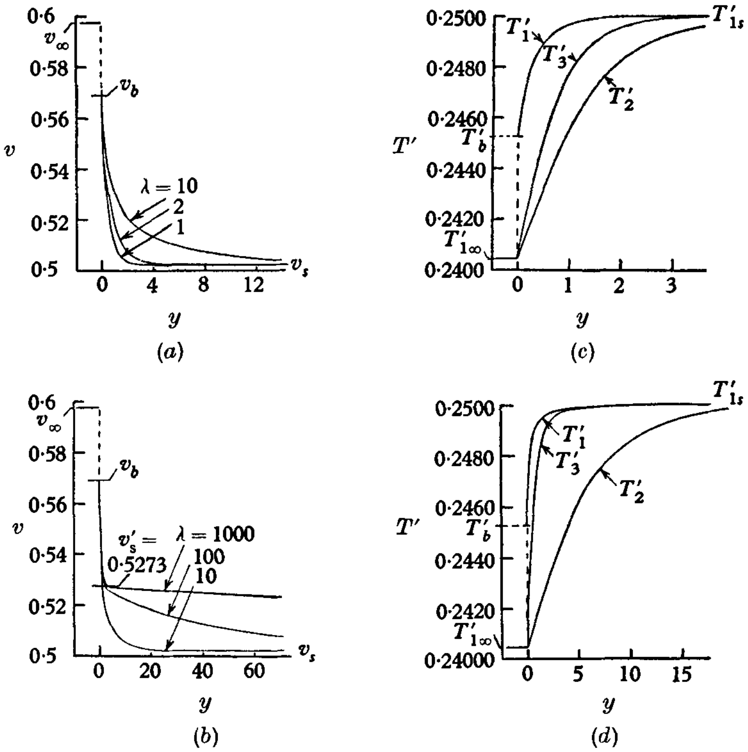

FIGURE 3. Partly dispersed wave. $M_{e\infty} = 1{\cdot}1$; $v_\infty = 0{\cdot}5977$, $v_s = 0{\cdot}5023$; $\gamma_1 = 7/5$, $\gamma_e = 11/9$. $\lambda = \tau_2/\tau_3$. (a) Velocity profiles for $\lambda = 1$, 2, 10. (b) Velocity profiles for $\lambda = 10$, 100, 1000. (c) Temperature profiles for $\lambda = 2$. (d) Temperature profiles for $\lambda = 10$.

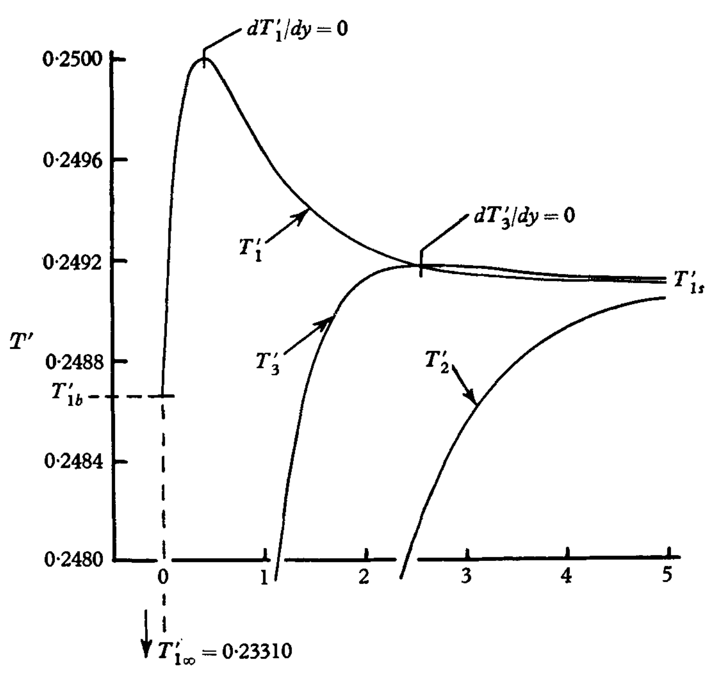

FIGURE 4. Partly dispersed wave. $M_{e\infty} = 1{\cdot}18$; $v_\infty = 0{\cdot}63, v_s = 0{\cdot}47; \gamma_1 = 7/5, \gamma_e = 11/9$. Temperature profiles for $\lambda = 2$, $\lambda = \tau_2/\tau_3$.

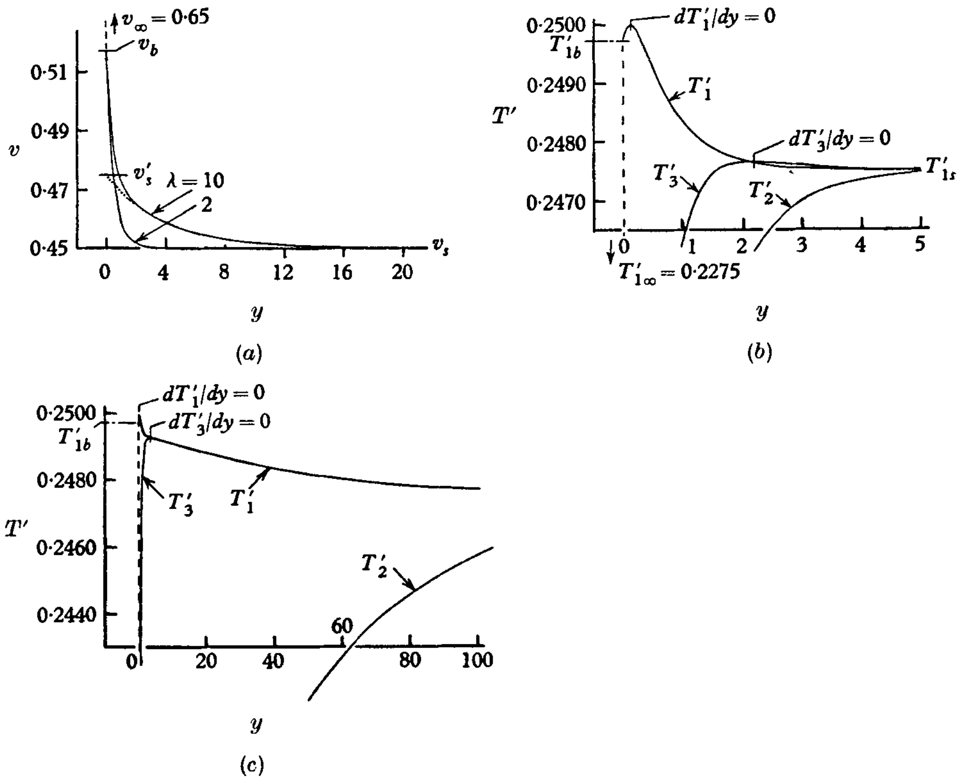

FIGURE 5. Partly dispersed wave. $M_{e\infty} = 1{\cdot}23$; $v_\infty = 0{\cdot}65, v_s = 0{\cdot}45$; $\gamma_1 = 7/5, \gamma_e = 11/9$. $\lambda = \tau_2/\tau_3$. (*a*) Velocity profiles for $\lambda = 2$, 10. (*b*) Temperature profiles for $\lambda = 2$. (*c*) Temperature profiles for $\lambda = 100$.

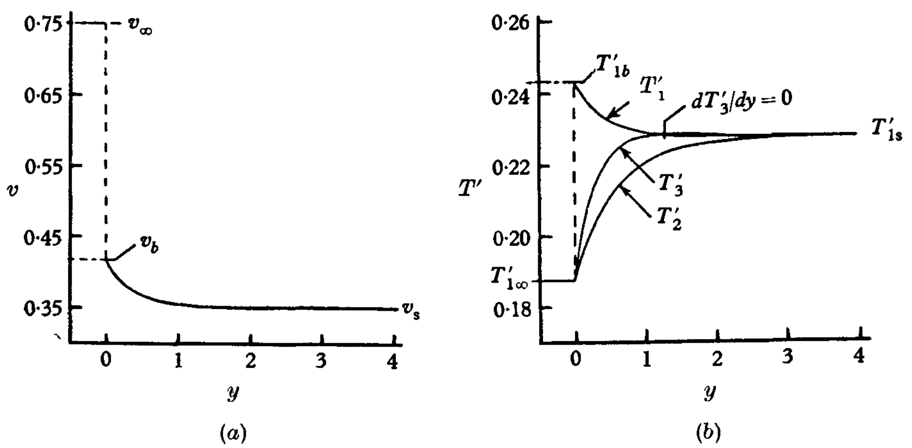

FIGURE 6. Partly dispersed wave. $M_{e\infty} = 1{\cdot}57$; $v_\infty = 0{\cdot}75$, $v_s = 0{\cdot}35$; $\gamma_1 = 7/5$, $\gamma_e = 11/9$. $\lambda = \tau_2/\tau_3$. (*a*) Velocity profiles for $\lambda = 2$. (*b*) Temperature profiles for $\lambda = 2$.

A further increase in the wave strength leads next to the state in which dT_1'/dy equal to zero still occurs after the discontinuity, but this time with $T_{1b}' > T_{1s}'$. Apart from this latter difference, figure 5 (*b*) is very similar to figure 4, both being for the case λ equal to 2. However, when λ equals 100 figure 5 (*c*) shows that T_3' quickly reaches a condition in which it is for all practical purposes equal to T_1'. The numerical results do show that T_3' exceeds T_1' from about $y = 3{\cdot}5$ onwards, but this does not show up on the scale of the figure. The peak in the T_1' curve occupies a very small region in the whole relaxation zone. Figure 5 (*a*) shows (on a magnified vertical scale, so that once again the whole wave amplitude is not exhibited) the variations of v for two values of λ. v_s' is found from equation (92) (mode 2 assumed frozen), and the dotted line represents the results of a one-mode calculation for the relaxation of mode 2 only from this point on. It can be seen that the exact two-mode and approximate one-mode calculations quickly merge for the chosen value of λ, namely 10. Whilst the one-mode approximation is undoubtedly good for v in this case, it is most important to note that since $v_s' < \frac{1}{2}$ the whole of the peak in T_1' and, of course, the overshoot in T_3' would be lost from the related approximate temperature curves.

Finally, figure 6 illustrates the behaviour of velocity and temperatures for a 'strong' wave, which has no maximum for T_1' in the relaxation zone. Figure 6 (*b*) indicates that T_3' exceeds T_1' for the particular value of λ chosen, namely 2, although the extent of the overshoot is clearly not large compared with the overall variations involved. The picture will now remain substantially the same for all waves of greater strength.

Although the results discussed above are for particular cases it is safe to draw some general conclusions from them. In particular, it is true that the velocity (and hence density and pressure) variations in the continuous regions of the waves do not respond significantly to the details of internal mode behaviour. Experimental investigations which set out to determine relaxation behaviour by measuring any of these quantities may therefore be expected to be somewhat insensitive. There seems to be a good case for pursuing the development of experimental techniques which could record directly those interesting variations in the temperatures which have been exemplified here.

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